§3.4 The Binomial Distribution

$\boxed{\text{Binomial}(n, p)}$

A Binomial R.V. is the sum of n independent Bernoulli R.V. (called "trials")

Recall: The standard example of a Bernoulli R.V. is flipping a coin

$$X = \begin{cases} 1 & \text{if Heads} \\ 0 & \text{if Tails} \end{cases}$$

$$X \sim \text{Bernoulli}(p = 1/2)$$

→ Extending this, the standard example of a Binomial R.V. is flipping many coins $X = \#\text{Heads}$

$$X \sim \text{Binomial}(n = \#\text{coins},\ p = 1/2)$$

(each individual coin flip is a "Bernoulli trial" X = sum of all of the trials)

Binomial random variables count the number of times something happens out of a given fixed, finite number of times the thing could have happened.

Example: A student attends $4/5$ of his classes.

Let $X = \#$(CVE 303 lectures student attends in one semester (out of 28 total))

Then $X \sim \text{Binomial}(n = 28,\ p = 4/5)$

---

p.m.f. & c.d.f. $X \sim \text{Binomial}(n, p)$ ← {X = #successes while performing n trials with $P(\text{success}) = p$}

$$f(x) = P(X = x)$$

$$= \underbrace{\binom{n}{x}}_{\substack{\text{\# of ways } x \text{ successes can}\\ \text{be distributed among } n \text{ trials}}} \cdot \underbrace{p^x}_{\substack{\text{Prob. of } x \text{ trials}\\ \text{being success}}} \cdot \underbrace{(1-p)^{n-x}}_{\substack{\text{Prob. of } (n-x)\\ \text{trials not being success}}}$$

$$F(x) = P(X \le x)$$

$$= P(X=0) + P(X=1) + \cdots + P(X=x)$$

$$= \binom{n}{0}p^0(1-p)^n + \binom{n}{1}p^1(1-p)^{n-1} + \cdots + \binom{n}{x}p^x(1-p)^{n-x}$$

[There unfortunately is not any simpler way to write or compute this:

$$F(x) = \sum_{k=0}^{x} \binom{n}{k} p^k (1-p)^{n-k}$$]

The name "Binomial" comes from the fact that the p.m.f. values $f(x) = \binom{n}{x}p^x(1-p)^{n-x}$ are the terms in the binomial expansion

$$(p + (1-p))^n = \binom{n}{0}p^0(1-p)^n + \binom{n}{1}p^1(1-p)^{n-1} + \cdots$$

For example if $n = 4$ and $p = 1/3$ ← $X \sim \text{Binom}(4, 1/3)$

$$(1/3 + 2/3)^4 = \underbrace{(2/3)^4}_{P(X=0)} + \underbrace{4(1/3)(2/3)^3}_{P(X=1)} + \underbrace{6(1/3)^2(2/3)^2}_{P(X=2)} + \cdots$$

Recall $\binom{n}{k}$ "n choose k" = # (ways to choose k items from n possibilities without replacement & without ordering)

These numbers are called "binomial coefficients"

They can also be computed using "Pascal's Triangle"

$$\binom{n}{k} = \frac{\overbrace{(n)\cdot(n-1)\cdot(n-2)\cdots(n-\cdots)}^{k \text{ numbers}}}{\underbrace{(1)\cdot(2)\cdot(3)\cdots(k)}_{k \text{ numbers}}}$$

This is often written "$\frac{n!}{k!\,(n-k)!}$"

Examples

$$\binom{10}{4} = \frac{10\cdot 9\cdot 8\cdot 7}{1\cdot 2\cdot 3\cdot 4} = 231$$

$$\binom{11}{7} = \frac{11\cdot 10\cdot 9\cdot 8\cdot \cancel{7\cdot 6\cdot 5}}{1\cdot 2\cdot 3\cdot 4\cdot \cancel{5\cdot 6\cdot 7}} = 330$$

$$\binom{8}{3} = \frac{8\cdot 7\cdot 6}{1\cdot 2\cdot 3} = 56$$

$$\binom{11}{4} = \frac{11\cdot 10\cdot 9\cdot 8}{1\cdot 2\cdot 3\cdot 4} = 330$$

Equal!!

Actually computing with binomial coefficients is annoying, so we'll just stick with using R:

> If $X \sim \text{Binomial}(n, p)$ then
>
> p.m.f. $f(x) = P(X = x)$ is
>
> **dbinom**(x, size = n, prob = p)
>
> c.d.f. $F(x) = P(X \le x)$ is
>
> **pbinom**(x, size = n, prob = p)
>
> (You can skip writing "size =" & "prob =" if you write in this order.)

Mean & Variance of Binomial

If $Y_k \sim \text{Bernoulli}(p)$ then

$$\underbrace{Y_1 + Y_2 + \cdots + Y_n}_{n \text{ times}} = X \sim \text{Binomial}(n, p)$$

Using linearity of expected value

$$E[X] = E[Y_1 + Y_2 + \cdots + Y_n] = n\,E[Y] = \underline{np}$$

A similar property of Variance of independent r.v. gives

$$\text{Var}[X] = \text{Var}[Y_1 + \cdots Y_n] = n\,\text{Var}[Y] = \underline{np(1-p)}$$

Examples: Suppose $X \sim \text{Binomial}(50, 1/3)$

(A) $P(X = 27)$

dbinom(27, 50, 1/3) $\approx .00126$

(B) $P(X \le 27)$

pbinom(27, 50, 1/3) $\approx .99918$

(C) $P(X \ge 27)$

$1 - P(X < 27) = 1 - P(X \le 26)$

$= 1 -$ pbinom(26, 50, 1/3)

$\approx .00208$

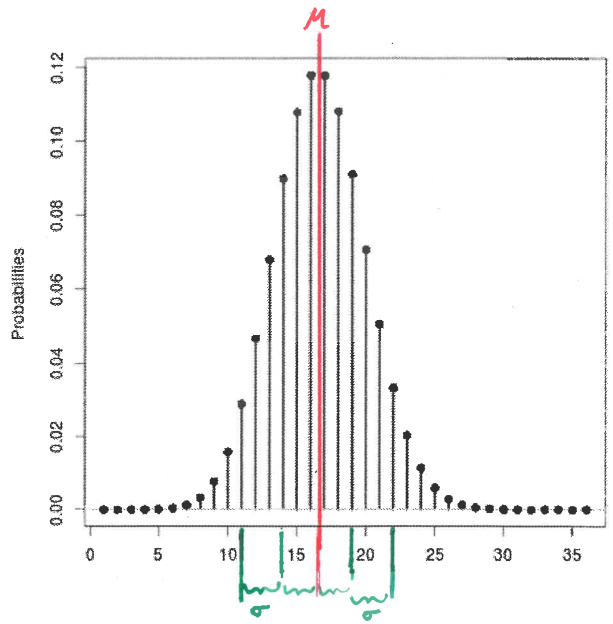
(D) $P(24 \le X \le 28)$

$P(X \le 28) - P(X \le 23)$

$=$ pbinom(28, 50, 1/3) $-$ pbinom(23, 50, 1/3) $\approx .02189$

pmf of $X \sim \text{Binomial}(50, 1/3)$ can be plotted with the commands

```
library(discreteRV)
plot(RV(0:50, dbinom(0:50, 50, 1/3)))
```



plotting this, you can see that probability is tightly concentrated around the mean

$$\mu = 50 \cdot 1/3 \approx 16.7$$

once you move one standard deviation away

$$\sigma = \sqrt{50 \cdot 1/3 \cdot 2/3} = \sqrt{100/9} = 10/3 \approx 3.3$$

the probability has dropped a lot.

→ $P(X \le 16.7 - 3.3) =$ pbinom(13, 50, 1/3) $\approx .1715$

→ $P(X \ge 16.7 + 3.3) = 1 -$ pbinom(20, 50, 1/3) $\approx .1259$

most probability is within one std. dev. of mean

$P(13 < X \le 20) =$ pbinom(20, 50, 1/3) − pbinom(13, 50, 1/3)

$\approx .7026$

and even more within two std. dev. of mean

$P(10 < X \le 23) =$ pbinom(23, 50, 1/3) − pbinom(10, 50, 1/3)

$\approx .9494$